On commutator subgroups of finite groups

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It is well known that in a group the product of two commutators needs not be a commutator; consequently the commutator subgroup cannot be defined as the set of all commutators, but only as the subgroup generated by those. In a previous paper [1] the author proved some theorems concerning groups having commutator subgroups whose elements are commutators. The aim of this paper is to give results in the same field. The theorems to be discussed will be called theorems I and II (see [1]).

THEOREM I: *A finite group in which all Sylow subgroups are cyclic has the property that all the elements of the commutator subgroup are commutators.*

THEOREM II: *Let G be a group of order n such that;*

1) *all the elements of its commutator subgroup are commutators;*
2) *the number of its elements of order 2 is less than $\frac{n}{2}$.*

Then all the elements of a certain coset modulo the commutator subgroup can be represented as a product of n distinct factors:

$$b = a_1 a_2 \cdots a_n\,; \quad b \in cK\,, \quad a_i \in G \qquad (i = 1, 2, \cdots, n)$$

$a_j \neq a_i$ if $i \neq j$ and cK is a coset mod K.

Several results concerning commutator subgroups may be found in [2], [3].

By theorem II close connection is established between the problem of L. Fuchs concerning commutator subgroups and the problem of finite groups in which the elements of the commutator subgroup are commutators.

The problem of L. Fuchs (see [1]) may be stated as follows: Let G be a finite group of order n and K its commutator subgroup. Since the elements of G commute mod K, every product of n distinct factors belongs to the same coset mod K. Can every element of this coset be represented as a product of n distinct factors?

LEMMA 1. *If in a finite group G of order n the number of elements of order 2 is $\frac{n}{2}$, then the elements of its commutator subgroup K are commutators.*

PROOF. In an arbitrary finite group of even order the number of the involutory elements is even, therefore the number of elements of order two in G assuming $\frac{n}{2}$, is odd. Thus n is even and not divisible by 4.

Let us denote by $b_1, b_2, \cdots, b_{\frac{n}{2}}$ the elements of order two, and by $a_1, a_2, \cdots, a_{\frac{n}{2}}$ the elements whose order is not two.

Let P be the regular representation of G. Then all the even permutations form a subgroup J in P. There correspond to each elements $b_1, b_2, \cdots, b_{\frac{n}{2}}$ in P such permutations which are the product of $\frac{n}{2}$ disjoint transpositions. Since all of these elements are odd permutations, none of them are contained in J. The elements

$$b_1b_1,\ b_1b_2,\ \cdots,\ b_1b_{\frac{n}{2}}$$

are all distinct and all of the corresponding permutations are even. In such a way the products in question are equal to the elements $a_1, a_2, \cdots, a_{\frac{n}{2}}$ i.e., the only elements, which are contained in J. Let us denote the commutator subgroup of P by K', then

$$J \supseteq K' .$$

The order of J is $\frac{n}{2}$, by the above arguments odd, so the order of its elements are odd. If $a_i^2=a_j^2$ $(i\neq j)$ would hold, then (if the order of a_i is $2m+1$ and the order of a_j is $2k+1$) by $a_i^{2m+1}=a_j^{2k+1}$, $a_ia_i^{2m}=a_j^{2k+1}=a_ia_j^{2m}$ holds hence $a_i=a_j^{2(k-m)+1}$ and $a_j^{2(k-m)}=a_ia_j^{-1}=a_i^{-1}a_j=a_j^{-2(k-m)}$ holds, which is a contradiction.

The elements

$$\begin{aligned}
a_1^2 &= b_1b_1b_1b_1 = b_1b_1b_1^{-1}b_1^{-1}\\
a_2^2 &= b_1b_2b_1b_2 = b_1b_2b_1^{-1}b_2^{-1}\\
&\vdots\\
a_{\frac{n}{2}}^2 &= b_1b_{\frac{n}{2}}b_1b_{\frac{n}{2}} = b_1b_{\frac{n}{2}}b_1^{-}b_{\frac{n}{2}}^{-1}
\end{aligned}$$

are all distinct and contained in J, thus they cover J, and so

$$J = K$$

and every element of K are commutators.

LEMMA 2. *The elements* $b_1, b_2, \cdots, b_{\frac{n}{2}}$ *are in a single conjugate class.*

PROOF. A group is called TJ group, if any two of its different 2-Sylow subgroups contain only the identity element in common. The TJ groups were studied by M. Suzuki in [6]. The elements of 2-Sylow subgroups are called 2-elements. A group is called 2-closed if the set of 2-elements forms a subgroup.

$b_1, b_2, \cdots, b_{\frac{n}{2}}$ belong to distinct 2-Sylow subgroup, more precisely the 2-Sylow subgroups of G may be characterized as cyclic groups of order two and their generators are b_i, $i=1, 2, \cdots, \frac{n}{2}$ $\left(\text{since } \frac{n}{2} \text{ is odd}\right)$. Thus G is a TJ group and G is not 2-closed as $b_1, b_2, \cdots, b_{\frac{n}{2}}$ and the identity element does not form a subgroup.

M. Suzuki proved (see [6]), that if a TJ group is not 2-closed, then the elements of order two form a single conjugated class. Using Suzuki's theorem, lemma 2. is immediate.

REMARK. Lemma 1. is an immediate consequence of lemma 2. We give it in a separate lemma because of its proof with the use of elementary arguments.

PROF. K. Honda kindly called my attention to the fact, that lemma 2. can be proved without using M. Suzuki's theorem.

A sharper result than theorem II will be proved.

THEOREM 1. *Let G be a group of order n such that:*

1) *all the elements of its commutator subgroup are commutators,*

2) *the number of its elements of order two is at most $\frac{n}{2}$.*

Then all the elements of a certain coset modulo the commutator subgroup can be represented as a product of n distinct factors:

$$b=a_1a_2 \cdots a_n\,; \quad b \in cK\,, \quad a_i \in G \qquad (i=1, 2, \cdots, n)$$

$a_j \neq a_l$ *if* $j \neq l$ *and* cK *is a coset modulo* K.

PROOF. We have only to prove our theorem in the case when the number of elements of order 2 in G is just $\frac{n}{2}$. Let $\prod_{i=1}^{\frac{n}{2}} b_i$ be denoted by c. Since $\frac{n}{2}$ is odd $c=\prod_{i=1}^{\frac{n}{2}} b_i = db_{\frac{n}{2}}$ where $d \in J$. Hence $c \notin J$, and thus c is an element of order two. Assume, that there exists an arbitrary element b_i of order two which commutes with a non involutory element a_j; $a_jb_i=b_ia_j$. Since $a_jb_i \notin J$, a_jb_i is of order two, and $a_jb_i=b_ia_j=b_ia_j^{-1}=a_j^{-1}b_i$. Hence $a_j=a_j^{-1}$, and this is however impossible.

By using trivial arguments, as a consequence of $c \notin J$ for arbitrary a_j, $a_jc \notin J$, $a_jca_j^{-1} \notin J$. In the case when $j \neq k$, in holds

$$a_j c a_j^{-1} = a_k c a_k^{-1}\ .$$

Using the notations $c = b_m$, $a_j^{-1} a_k = a_l$,

$$b_m a_l = a_l b_m$$

which is a contradiction.

Thus, for arbitrary b_i $\left(i=1, 2, \cdots, \frac{n}{2}\right)$ element of order two, there exists such an element a_j whose order is not two that

$$b_i = a_j c a_j^{-1}$$

holds. Thus $b_i = a_j c a_j^{-1} \prod_{k \neq j} a_k a_k^{-1}$.

REMARK. Prof. M. Suzuki has kindly informed me, that if G denotes a finite group of order n generated by an Abelian subgroup A of order m and an element t of order 2 satisfying the following relations: if $a \in A$ then $t^{-1}at = a^{-1}$, then every element of G not in A is an involution. So G contains exactly $\frac{n}{2}$ involutions if m is odd, and G has more than $\frac{n}{2}$ involutions if m is even, (see [7]). Thus theorem 1. may be generalized by pointing out its validity for the later case.

REMARKS. Professor I. D. MacDonald proved that for any given positive integer n there is a group G of finite order such that its commutator subgroup is cyclic; and it is not generated by any set of less than n commutators. The proof of this theory is almost the same as one of the examples which had been given by the same author in his paper (On cyclic commutator subgroups. Journal London Math. Soc. 38. (1963) 419–422).

Dr. A. H. Rhemtulla gave an example of finite group whose Sylow 2-subgroups are cyclic and its commutator subgroup contains elements which are not commutator themselves. The example is a nilpotent group of class two generated by n elements $x_1, x_2, \cdots, x_n$, such that $x_i^p = 1$ for all i and p is a prime greater than two.

The author is indebted to prof. I. D. MacDonald and A. H. Rhemtulla for their examples which show that certain generalizations of the result contained in this paper may not exist.

In a finite group G the subgroup generated by the commutators $gxg^{-1}x^{-1}$ as x takes all values in G and g is a fixed element of G denoted by $K(g)$ (Several properties of $K(g)$ may be found in [5]).

Theorem 2. *In a finite metabelian group G the subgroup $K(g)$ for any fixed g contains only commutators of the form $gxg^{-1}x^{-1}$.*

PROOF. MacDonald proved that every element of $K(g)$ is a commutator of the form $gxg^{-1}x^{-1}$ if and only if for arbitrary elements x, $y \in G$ there exists element $z \in G$ such that $xgx^{-1}g^{-1} \cdot gxyg^{-1}y^{-1}x^{-1} = gzg^{-1}z^{-1}$ holds. If G is metabelian then

$$[x, g][g, xy] = [x, g][g, x][g, y] = [g, y]$$

(see [4]), holds and the result is immediate.

The author is indebted to Prof. K. Honda for his valuable remarks.

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